The paradox of instantaneous rates of change

Differential calculus is about rates of change at an instant.

Integral calculus is about the total amount of change.

If a car travels 60 miles in 2 hours, the rate of change of the distance it has covered with respect to passing time is

$$\frac{60 \times \text{mile}}{2 \times \text{hour}} = 30 \text{ miles per hour.}$$

This is **not** an instantaneous rate of change; it is an average rate of change over two hours. A car's speed typically varies during that time.

How can there be such a thing as the car's speed at a particular instant? If a car is moving at 30 miles per hour, the speed at which it is moving would appear to be

$$\frac{0 \times \text{mile}}{0 \times \text{hour}}.$$

And if its speed is 72 miles per hour, it is still $\dfrac{0 \times \text{mile}}{0 \times \text{hour}}$.

Thus "0/0", "zero over zero", plays a central role in differential calculus.

The expression " $\dfrac{5}{0}$ " is undefined because one cannot fill in this blank:

$$0 \times \big(\text{what?}\big) = 5.$$

But " $\dfrac{0}{0}$ " is undefined for a different reason:

$$0 \times \big(\text{what?}\big) = 0.$$

This blank <u>can</u> be filled in. The problem is that that we cannot single out just one number that fits there. Any number at all makes the statement true. Thus

$$\begin{aligned} &\big(30 \text{ miles per hour}\big) \times \big(0 \text{ hours}\big) = 0 \text{ miles} \\ \text{and } &\underbrace{\big(72 \text{ miles per hour}\big)}_{\text{rate}} \times \underbrace{\big(0 \text{ hours}\big)}_{\text{time}} = \underbrace{0 \text{ miles}}_{\text{distance}} \\ &\text{etc.} \end{aligned}$$

Thus it might appear that we can make no sense of speed at a particular instant. Nonetheless we can say that the speedometer points at 72 miles per hour at a particular instant, and that that particular speedometer is accurate. If we cannot figure out how to make sense of a rate of change at an instant, then there can be no differential calculus.

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Here is one way to resolve the paradox. I suspect this particular way of doing it is easier to understand than the more usual one. Whether it is easier to work with later I don't know.

FIRST, we know what it means to say that a car is moving at a constant speed. 60 miles in 2 hours is $\dfrac{60}{2} = 30$ miles per hour, but if the speed is constant we could have used any other amount of time besides 2 hours and arrived at the same result. For example, observing the car for 1 second, we have

$$\begin{aligned}
&\frac{44 \times \text{foot}}{1 \times \text{second}} \\
&= \frac{44 \times \text{foot}}{1 \times \text{second}} \times 1 \times 1 \\
&= \frac{44 \times \text{foot}}{1 \times \text{second}} \times \frac{1 \times \text{mile}}{5280 \times \text{foor}} \times \frac{3600 \times \text{second}}{1 \times \text{hour}} \\
&= \frac{44 \times \cancel{\text{foot}}}{1 \times \cancel{\text{second}}} \times \frac{1 \times \text{mile}}{5280 \times \cancel{\text{foot}}} \times \frac{3600 \times \cancel{\text{second}}}{1 \times \text{hour}} \\
&= \frac{44 \times 3600}{5208} \times \frac{\text{mile}}{\text{hour}} \\
&= 30 \times \frac{\text{mile}}{\text{hour}} = 30 \text{ miles per hour.}
\end{aligned}$$

So much for <u>constant</u> speed.

SECOND, if another car, moving at a possibly varying speed, overtakes the car moving at a constant 30 miles per hour at a particular instant, that does **not** mean the other car is moving faster than 30 miles per hour at that instant. Here is why: The other car may be catching up to the constant-speed car, gaining on it from behind, necessarily going faster than 30 miles per hour, but gradually slowing down. Just at the instant when it catches up to the constant-speed car, its speed gets down to 30 miles per hour. But then it begins to move faster again, overtaking the constant-speed car. **However**, the fact that our car overtakes the car that moves at a constant speed of 30 miles per hour **does** mean that our car **is not moving more slowly than 30 miles per hour at that instant**.

So

- If at a particular instant our car overtakes another car moving at constant speed, then our car is not moving more slowly than that other car at that instant;
- If at a particular instant our car is overtaken by another car moving at constant speed, then our car is not moving more slowly than that other car at that instant.

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Suppose we find that

- At a particular instant we are not moving faster than any car moving at constant speed of 30 miles per hour or more; and
- At that particular instant we are not moving more slowly than any car moving at constant speed of 30 miles per hour or less.

Then our speed at that instant is exactly 30 miles per hour.

The paradox is resolved: It is possible to speak of the rate of change at a particular instant, despite the seeming fact that our speed is 0 miles/0 hours and we are facing the undefined expression 0/0, zero over zero.